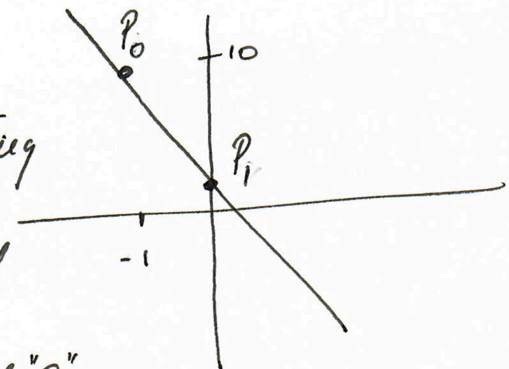
3.1 Interpolation and the Lagrange Polynomials.

(I) Lines

Consider the points: $\overset{(x_0,\, y_0)}{P_1(-1,10)}$ and $\overset{(x_1,\, y_1)}{P_2(0,3)}$



The line interpolating them can be obtained by determining "a_0" and "a_1" from $P(x) = a_0 + a_1 x$

or

$$\left.\begin{aligned} a_0 - a_1 &= 10 \\ a_0 + 0 &= 3 \end{aligned}\right\} \Rightarrow \quad a_0 = 3,\quad a_1 = -7 \qquad \boxed{P(x) = 3 - 7x} \qquad (1)$$

We can represent the same line as

$$Q(x) = \frac{x-0}{-1-0} * 10 + \frac{x-(-1)}{0-(-1)} * 3$$

$$\therefore \quad \boxed{Q(x) = -10x + 3(x+1).} \qquad (2)$$

Are (1) and (2) different polynomials?

In general, $\boxed{Q(x) = \dfrac{x - x_1}{x_0 - x_1}\, y_0 + \dfrac{x - x_0}{x_1 - x_0}\, y_1}$

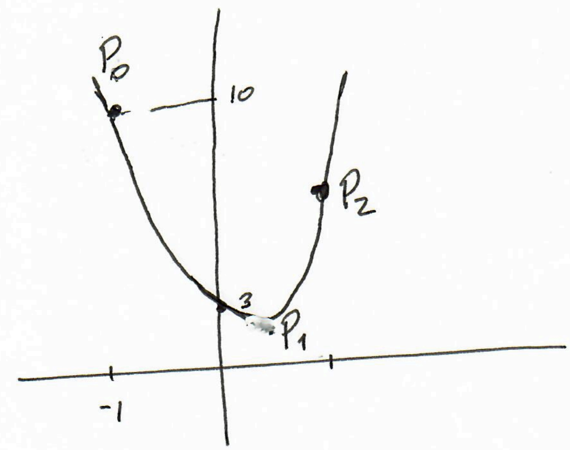
Is it possible to interpolate the two points given above with a polynomial degree two or higher?

(II) Parabolas. Second order polynomials

Consider the points:

P_0: $(x_0, y_0) = (-1, 10)$, P_1: $(x_1, y_1) = (0, 3)$, P_2: $(x_2, y_2) = (1, 6)$

These three points can be interpolated by a polynomial degree two



In fact,

$$P(x) = a_0 + a_1 x + a_2 x^2$$

interpolates the three points for appropriate a_0, a_1, a_2.

How can you find them?

$$\left.\begin{aligned} a_0 + a_1 + a_2 &= 10 \\ a_0 &= 3 \\ a_0 + a_1 + a_2 &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} a_0 &= 3 \\ a_1 &= -2 \\ a_2 &= 5. \end{aligned}$$

$$\therefore \quad \boxed{P(x) = 3 - 2x + 5x^2} \qquad (3)$$

Another possibility: use formula

$$Q(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Then,

$$Q(x) = \frac{x(x-1)}{2} * 10 + \frac{(x+1)(x-1)}{-1} * 3 + \frac{(x+1)x}{2} * 6$$

or

$$\boxed{Q(x) = 5x(x-1) - 3(x+1)(x-1) + 3(x+1)x} \qquad (4)$$

Are they (3) and (4) different polynomials?

How many polynomials can we use to interpolate these three points?

How many of them are of degree two or less?

Existence and Uniqueness of Interpolating Polynomials.

Example: Consider the three points

$(x_0, f(x_0)) = (-1, 10)$, $(x_1, f(x_1)) = (0, 3)$, $(x_2, f(x_2)) = (1, 6)$

Polynomials of degree at most two interpolating these three points are given by

① $P_2(x) = 3 - 2x + 5x^2$

② $Q_2(x) = 5x(x-1) - 3(x-1)(x+1) + 3(x+1)x$

③ $R_2(x) = 10 - 7(x+1) + 5(x+1)x$

④ $S_2(x) = 6 + 8(x-1) + 5(x-1)^2$

All of them are different representations of the same quadratic polynomial defined in ① by $P_2(x)$.

thm:- ① If $x_0, x_1, .., x_n$ are $(n+1)$ distinct numbers

② If $y_0, y_1, .., y_n$ are arbitrary real values.

$\Rightarrow$ there is a unique polynomial $p(x)$ of degree at most n such that

$$p(x_i) = y_i, \quad 0 \le i \le n.$$

<u>Proof.</u>– We can prove the existence ~~constructing~~ a Lagrange interpolating polynomials that interpolates the given data.

$$p_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}\, y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}\, y_1 +$$

Use this part as a definition of Lagrange Polynomials

$$+ \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\, y_i +$$

$$+ \cdots + \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}\, y_n$$

$$= \prod_{j=1}^{n}\left(\frac{(x-x_j)}{(x_0-x_j)}\right) y_0 + \prod_{\substack{j=0\\ j\neq 1}}^{n}\left(\frac{x-x_j}{x_1-x_j}\right) y_1 + \cdots$$

$$+ \cdots + \prod_{\substack{j=0\\ j\neq i}}^{n}\left(\frac{x-x_j}{x_i-x_j}\right) y_i + \cdots + \prod_{\substack{j=0\\ j\neq n}}^{n}\left(\frac{x-x_j}{x_n-x_j}\right) y_n$$

$$= l_0(x)\, y_0 + l_1(x)\, y_1 + \cdots + l_i(x)\, y_i + \cdots + l_n(x)\, y_n$$

or

$$p_n(x) = \sum_{i=0}^{n} l_i(x)\, y_i$$

To prove uniqueness, we note that

$$p_n(x_k) = y_k, \qquad k = 0, 1, .., n$$

And also $p_n(x)$ is a polyn. of degree at most n.

If there were another polyn. $Q(x)$ of degree at most n.

Such that

$$Q(x_k) = p_n(x_k) = y_k, \qquad k = 0, 1, .., n$$

then according to our previous Corollary

$$Q(x) = p_n(x), \quad \text{for all } x,$$

Since they agree in $n+1$ distinct numbers: $x_0, .., x_n$.

And are of degree at most n.

Back to our example:

$P_2(x)$, $Q_2(x)$, $R_2(x)$ and $S_2(x)$ are polynomials of degree two, and all of them interpolate $(-1, 10)$, $(0, 3)$ and $(1, 6)$. then, they represent the same polynomial of degree two.

$Q_2(x)$ is Lagrange form

$R_2(x)$ is in Newton form

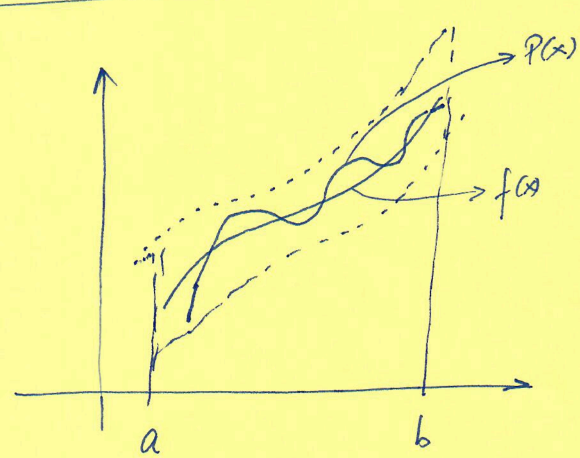
Approximation of $f(x) \in C[a,b]$ by polynomials.

<u>Thm</u>: (Weierstrass)

- $f \in C[a,b]$

then for all $\varepsilon > 0$, there exists $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon, \quad \text{for all } x \in [a,b].$$



As ε gets smaller, we may need to increase the degree of the polynomial.

Therefore, we would like to have an easy way to construct a polynomial of higher degree from another one of lower degree.

Comparing Taylor Polynomials with Lagrange Polynomials

<u>MAPLE</u>

This bound is determine from

$$\left| \frac{f'''(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right| = |Error| = |f(2.1) - P(2.1)|$$

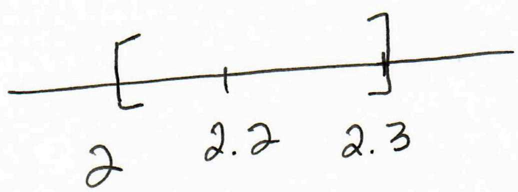
Now, $\quad f'''(\xi) = \frac{2}{\xi^3}$

The points considered for interpolation are

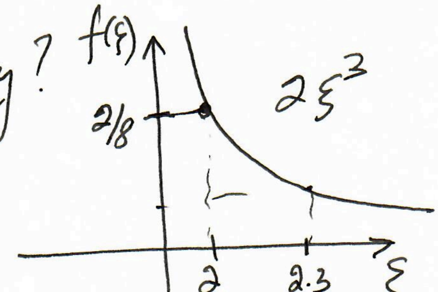
$x_0 = 2.$

$x_1 = 2.2$

$x_2 = 2.3$



Inside the interval $[2, 2.3]$ $|f'''(\xi)|$ reaches its maximum at $\xi = 2$. why?



$$\Rightarrow |f''(\xi)| = \left|\frac{2}{\xi^3}\right| \leq \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

$$\Rightarrow |Error| = \left| \frac{f''(\xi)}{3!} \overset{(2.1-2)}{(x-x_0)} \overset{(2.1-2.2)}{(x-x_1)} \overset{(2.1\ 2.3)}{(x-x_2)} \right| = \left|\frac{2}{\xi^3}\right| \frac{|0.1 * (-0.1) * (-0.2)|}{3!}$$

$$\leq \frac{1}{4} * \frac{1}{3!} * 2 \times 10^{-3} = \frac{2 \times 10^{-3}}{24} = 8.3 \times 10^{-5}$$

Proof.– If $x = x_i$ (node) $i = 0, .., n$ is trivial.

If $x \neq x_i$, $(i = 0, 1, .., n)$ and is fixed

We will define

$$\omega(t) = \prod_{i=0}^{n} (t - x_i)$$ polyn. of degree $n+1$ in the variable t.

$$c = \frac{f(x) - p_n(x)}{\omega(x)},$$ Constant for x fixed. well defined because $\omega(x) \neq 0$.

$$\boxed{\varphi(t) = f(t) - p_n(t) - c\,\omega(t)}$$ function in the variable t.

Clearly, $$\varphi(x_j) = f(x_j) - p_n(x_j) - c\,\omega(x_j) =$$
$$= f(x_i) - f(x_i) - c\,(x_j - x_1) \cdots \overset{0}{\cancel{(x_j - x_j)}} \cdots (x_j - x_n)$$
$$= 0. \quad \text{for all } j = 0, .., n.$$

Also, $$\varphi(x) = f(x) - p_n(x) - c\,\omega(x) =$$
$$= f(x) - p_n(x) - \left(\frac{f_n(x) - p_n(x)}{\cancel{\omega(x)}} \right) \cancel{\omega(x)}$$
$$= 0.$$

Therefore, $\varphi(t)$ has $n+2$ zeros

$x_0, x_1, .., x_n$ and x.

<u>Rolle's Thm:</u> ① $f \in C[a,b]$

② $f'(x)$ exists on (a,b)

③ $f(a) = f(b)$

$\Rightarrow$ there exist $c \in (a,b)$ such that

$$f'(c) = 0.$$

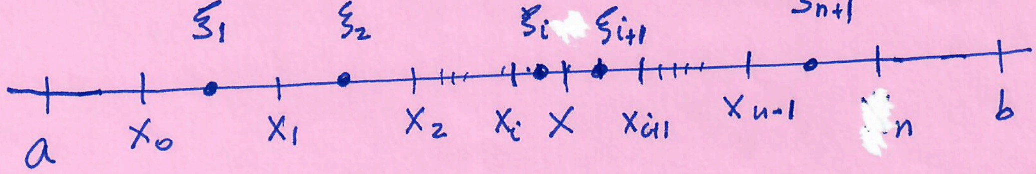
Our function $\varphi(t)$ satisfies:

$\varphi(t) \in C^{n+1}[a,b]$, because $f \in C^{n+1}[a,b]$ and $p_n(t)$ and $\omega(t)$ are polynomials $C^{\infty}[a,b]$.

Also, $\varphi(x_0) = \varphi(x_1) = \cdots = \varphi(x_n) = \varphi(x) = 0$

So according to Rolle's thm, between any two of this consecutive roots there is a ξ such that

$$\varphi'(\xi) = 0$$



Therefore, $\varphi'(t)$ has at least $n+1$ roots in $[a,b]$.

Similarly $\varphi''(t)$ " " " n roots " "

$\vdots$

$\varphi^{(n+1)}(t)$ " " " $(n+2)-(n+1) = 1$ root

Let $\hat{\xi}$ be a root of $\varphi^{(n+1)}(t)$. in (a,b)

Now,

$$\varphi^{(n+1)}(t) = f^{(n+1)}(t) - \cancel{p_n^{(n+1)}(t)}^{\,0} - c\,\omega^{(n+1)}(t)$$

$$= f^{(n+1)}(t) - c\,(n+1)!$$

therefore, $\varphi^{(n+1)}(\hat{\xi}) = f^{(n+1)}(\hat{\xi}) - c\,(n+1)! = 0$

$$\Rightarrow \quad f^{(n+1)}(\hat{\xi}) - \frac{f(x) - p_n(x)}{\omega(x)}(n+1)! = 0$$

$$\Rightarrow \quad f(x) - p_n(x) = \frac{\omega(x)}{(n+1)!} f^{(n+1)}(\hat{\xi})$$

or $$\boxed{f(x) = p_n(x) + \frac{f^{(n+1)}(\hat{\xi})}{(n+1)!} \prod_{i=0}^{n} (x - x_i)} \quad (*)$$

<u>Thm.</u> (Bound on the interpolation error)

① $f \in C^{n+1}[a,b]$

② $|f^{(n+1)}(x)| \leq M$, for all $x \in [a,b]$

③ $p_n(x)$ polyn. of degree n interpolating $f(x)$ at $(n+1)$ equally spaced nodes. in $[a,b]$ including "a" and "b".

$$\Rightarrow \quad |f(x) - p(x)| \leq \frac{1}{4(n+1)} M \left(\frac{b-a}{n}\right)^{n+1}$$

<u>Proof.</u>- It's based in the inequality

$$\prod_{i=0}^{n} |x-x_i| \leq \frac{1}{4} h^{n+1} n! \qquad (**)$$

(See book for a proof.)

From (*), we have for $\xi \in (a,b)$ that

$$|f(x)-p(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \left| \prod_{i=0}^{n} (x-x_i) \right| \leq$$

$$\leq \frac{M}{(n+1)!} \prod_{i=0}^{n} |x-x_i| \overset{(**)}{\leq} \frac{M}{(n+1)!} \frac{1}{4} h^{n+1} n!$$

$$= \frac{M}{n+1} \frac{1}{4} \left(\frac{b-a}{n}\right)^{n+1}$$

Since $h = \frac{b-a}{n}$.